The evolution operator technique in solving the Schrodinger equation, and its application to disentangling exponential operators and solving the problem of a mass-varying harmonic oscillator

1988 J. Phys. A: Math. Gen. 21 4115

(http://iopscience.iop.org/0305-4470/21/22/015)

J. Phys. A: Math. Gen. 21 (1988) 4115-4131. Printed in the UK

The evolution operator technique in solving the Schrödinger equation, and its application to disentangling exponential operators and solving the problem of a mass-varying harmonic oscillator

C M Cheng and P C W Fung

Physics Department, University of Hong Kong, Bonham Road, Hong Kong

Received 10 May 1988, in final form 15 June 1988

Abstract. We present a method for finding the evolution operator for the Schrödinger equation for the Hamiltonian expressible as

$$\hat{H}(t)=a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_-$$

where $\hat{J}_+, \hat{J}_0$ and $\hat{J}_-$ are the SU(2) group generators. Such a method is applied to the disentangling technique for exponential operators which are not necessarily unitary. As a demonstration of our general approach, we solved the problem of a harmonic oscillator with a varying mass.

1. Introduction

The evolution operator method has long been used to solve problems in quantum mechanics. This method was first proposed by Dyson [1] and recently developed further by Lam and Fung [2]. But finding the explicit form of an evolution operator for the evolution equation

$$i\hbar\frac{\partial\hat{U}(t)}{\partial t}=\hat{H}(t)\hat{U}(t) \qquad \hat{U}(0)=1 \tag{1.1}$$

is not an easy task. However, if the Hamiltonian takes the following form:

$$\hat{H}(t)=\sum_{i=1}^{m} a_i(t)\hat{H}_i \tag{1.2}$$

where $a_i(t)$ are functions of time, and $\{\hat{H}_i, i=1,\ldots,m\}$ forms a closed Lie algebra $\mathscr{L}$ (of dimension m), then the evolution operator can be expressed locally in any one of the following forms [3, 4]:

$$\hat{U}(t)=\exp\left(\sum_{i=1}^{m} b_i(t)\hat{H}_i\right) \tag{1.3}$$

$$\hat{U}(t)=\prod_{i=1}^{m}\exp[c_i(t)\hat{H}_i] \tag{1.4}$$

where $b_i(t)$, $c_i(t)$ are functions of time. With the above relations, the evolution equation (1.1) can be considered as solved if the explicit expressions for $b_i(t)$ or $c_i(t)$ are found. The above procedure provides a way of finding the evolution operator for the Hamiltonian (1.2).

In this paper we shall concentrate on a particular time-dependent Hamiltonian which comprises SU(2) group generators

$$\hat{H}(t)=a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_- \tag{1.5}$$

where $\hat{J}_+$, $\hat{J}_0$ and $\hat{J}_-$ form the SU(2) Lie algebra:

$$[\hat{J}_+, \hat{J}_-]=2\hat{J}_0 \tag{1.6a}$$

$$[\hat{J}_0, \hat{J}_\pm]=\pm\hat{J}_\pm \tag{1.6b}$$

and $a_i(t)$ are arbitrary functions of time. We shall employ the particular choice of evolution operator (1.4) and the evolution equation (1.1) in § 2 to obtain a set of ordinary differential equations for $c_i(t)$. Once $a_i(t)$ are given explicitly, the set of ordinary differential equations can be solved and we can obtain the expression for the evolution operator. In this way the problem of solving the evolution equation, which is an operator equation, is reduced to solving a set of ordinary differential equations. This is the key result we obtain. The reason for choosing the particular form (1.4) for the evolution operator will be discussed in § 2.

As a first application, we shall consider an exponential operator in § 3 whose exponent is a linear combination of SU(2) group generators, namely $\hat{J}_\pm, \hat{J}_0$. By employing the result in § 2, we shall devise a method to disentangle this exponential operator into a product of exponential operators. This procedure provides a way to uncouple exponential operators which are not necessarily unitary. As a particular example, we shall derive the well known Baker-Campbell-Hausdorff formula [5].

In § 4, we shall apply the result in § 2 to the problem of a time-dependent harmonic oscillator with a varying mass. We find the evolution operator explicitly. The wavefunction for the evolution of an initially coherent state is then obtained. Hence the coherence and squeezing properties of this wavefunction can be studied. Besides the expectation values for the energy, the position and the momentum are found. It is the first time, as the authors realise, that these expectation values are found explicitly by the direct use of the evolution operator method. These results show an intimate relationship between the mass-varying harmonic oscillator we consider here and the classical damped harmonic oscillator. More detailed discussion is given in this section.

Section 5 concludes this investigation.

2. Evolution operator

Firstly we start with the Hamiltonian (1.5)

$$\hat{H}(t)=a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_-.$$

The Schrödinger equation corresponding to this Hamiltonian is

$$\hat{H}(t)|\Phi(t)\rangle = \mathrm{i}\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle. \tag{2.1}$$

As usual, the evolution operator is introduced as follows:

$$|\Phi(t)\rangle = \hat{U}(t,0)|\Phi(0)\rangle \tag{2.2}$$

where $|\Phi(0)\rangle$ is the wavefunction at time $t=0$. Inserting (2.2) into (2.1) produces the evolution equation

$$\hat{H}(t)\,\hat{U}(t,0)=\mathrm{i}\hbar\frac{\partial}{\partial t}\,\hat{U}(t,0) \qquad \hat{U}(0,0)=1. \tag{2.3}$$

Since $\hat{J}_0$, $\hat{J}_\pm$ form a closed Lie algebra su(2), the evolution operator can be expressed in the following form:

$$\hat{U}(t,0)=\exp(c_1(t)\hat{J}_+)\exp(c_2(t)\hat{J}_0)\exp(c_3(t)\hat{J}_-) \tag{2.4}$$

where $c_i(t)$ are to be determined. We have chosen this particular form for $\hat{U}(t,0)$ because it is expressed as a product of exponential operators, and direct differentiation with respect to time for this operator can be readily carried out. Now we have

$$\frac{\partial}{\partial t}\hat{U}(t,0)=\{[\dot{c}_1-c_1\dot{c}_2-c_1^2\exp(-c_2)\dot{c}_3]\hat{J}_+ +[\dot{c}_2+2c_1\exp(-c_2)\dot{c}_3]\hat{J}_0$$
$$+\exp(-c_2)\dot{c}_3\hat{J}_-\}\hat{U}(t,0). \tag{2.5}$$

Putting the above result together with (2.2) into (2.1), and comparing the two sides, we obtain three ordinary differential equations:

$$\mathrm{i}\hbar[\dot{c}_1-c_1\dot{c}_2-c_1^2\exp(-c_2)\dot{c}_3]=a_1 \tag{2.6a}$$

$$\mathrm{i}\hbar[\dot{c}_2+2c_1\exp(-c_2)\dot{c}_3]=a_2 \tag{2.6b}$$

$$\mathrm{i}\hbar\exp(-c_2)\dot{c}_3=a_3 \tag{2.6c}$$

which can be rewritten as

$$\dot{c}_1=a_1'+a_2'c_1-a_3'c_1^2 \tag{2.7a}$$

$$\dot{c}_2=a_2'-2a_3'c_1 \tag{2.7b}$$

$$\dot{c}_3=a_3'\exp(c_2) \tag{2.7c}$$

with the initial conditions

$$c_1(0)=0 \tag{2.8a}$$

$$c_2(0)=0 \tag{2.8b}$$

$$c_3(0)=0. \tag{2.8c}$$

The a_j' are given by

$$a_j'=a_j/\mathrm{i}\hbar. \tag{2.9}$$

Note that we have written $c_1(t)$, etc, as c_1 for simplicity with the understanding that they are explicitly time dependent. Equation system (2.7) is our main result, which gives the relationship between c_i and a_i. It should be noted that equation (2.7a), which is just the Riccati equation, is the key equation we have to solve first. Once it is solved, the other two equations can be solved readily to give

$$c_2=\int_0^t \mathrm{d}u(a_2'-2a_3'c_1) \tag{2.10}$$

$$c_3=\int_0^t \mathrm{d}u\, a_3'\exp(c_2). \tag{2.11}$$

3. The disentangling technique

Very often, one encounters the problem of disentangling exponential operators into products of exponential operators. One formula that is often employed to serve such a purpose is the Baker-Campbell-Hausdorff formula for the su(2) Lie algebra [5]:

$$\exp(\alpha\hat{J}_+ - \alpha^*\hat{J}_-) = \exp(\beta_1\hat{J}_+)\exp(\beta_2\hat{J}_0)\exp(\beta_3\hat{J}_-) \tag{3.1}$$

where

$$\alpha = -\tfrac{1}{2}\vartheta\exp(-\mathrm{i}\varphi) \tag{3.2}$$

$$\beta_1 = -\exp(-\mathrm{i}\varphi)\tan\tfrac{1}{2}\vartheta \tag{3.3}$$

$$\beta_2 = -2\ln(\cos\tfrac{1}{2}\vartheta) \tag{3.4}$$

$$\beta_3 = -\beta_1^*. \tag{3.5}$$

Recently two papers have been published on the disentangling technique [6, 7]. In particular, Truax [6] worked out explicitly the Baker-Campbell-Hausdorff formula for the groups SU(2) and SU(1, 1), and the operators involved are unitary.

In the following we shall give an alternative method for uncoupling exponential operators of the following type:

$$\hat{S}_1 \equiv \exp(a_1\hat{J}_+ + a_2\hat{J}_0 + a_3\hat{J}_-) \tag{3.6}$$

where a_1, a_2 and a_3 are coefficients. This operator is not necessarily unitary. To start with, we introduce a parameter λ in $\hat{S}_1$ as follows:

$$\hat{S}_1(\lambda) = \exp[\lambda(a_1\hat{J}_+ + a_2\hat{J}_0 + a_3\hat{J}_-)]. \tag{3.7}$$

Differentiating with respect to λ gives the following differential equations:

$$\frac{\partial}{\partial\lambda}\hat{S}_1(\lambda) = \mathfrak{H}\hat{S}_1(\lambda) \qquad \hat{S}_1(0) = 1 \tag{3.8}$$

where

$$\mathfrak{H} = a_1\hat{J}_+ + a_2\hat{J}_0 + a_3\hat{J}_-. \tag{3.9}$$

Equation (3.8) is similar in form to the evolution equation (1.1). Thus the former result in § 2 can be employed. To uncouple the exponential operator (3.7), we suppose that $\hat{S}_1(\lambda)$ can be expressed in the following form:

$$\hat{S}_1(\lambda) = \hat{S}_2(\lambda) \tag{3.10}$$

where

$$\hat{S}_2(\lambda) = \exp(c_1(\lambda)\hat{J}_+)\exp(c_2(\lambda)\hat{J}_0)\exp(c_3(\lambda)\hat{J}_-) \tag{3.11}$$

and $c_i(\lambda)$ are to be determined. Since $\hat{S}_2(\lambda)$ is also a solution to (3.8), we can simply put (3.11) back into (3.8) and obtain the differential equations for $c_i(\lambda)$:

$$\frac{\partial}{\partial\lambda}c_1 = a_1 + a_2c_1 - a_3c_1^2 \tag{3.12}$$

$$\frac{\partial}{\partial\lambda}c_2 = a_2 - 2a_3c_1 \tag{3.13}$$

$$\frac{\partial}{\partial\lambda}c_3 = a_3\exp(c_2) \tag{3.14}$$

with the initial conditions

$$c_1(0)=0 \tag{3.15a}$$

$$c_2(0)=0 \tag{3.15b}$$

$$c_3(0)=0. \tag{3.15c}$$

Thus if (3.12)-(3.14) are solved, the disentangling process of the exponential operator in (3.10) is established. To proceed further, we make the following restriction that both a_2 and $a_1 a_3$ are real quantities; such a restriction† will simplify the mathematics, and it is satisfied in our subsequent derivation of the Baker-Campbell-Hausdorff formula later in this section.

By making the following transformation

$$c_1=\frac{\partial u}{\partial \lambda}(a_3 u)^{-1} \tag{3.16}$$

the non-linear equation (3.12) is transformed to a second-order linear differential equation

$$\frac{\partial^2 u}{\partial \lambda^2}-a_2\frac{\partial u}{\partial \lambda}-a_1 a_3 u=0. \tag{3.17}$$

Since a_2, $a_1 a_3$ are, as we considered above, real, the above equation is a real linear equation which can be solved readily. As usual we consider the following characteristic equation for (3.17):

$$\delta^2-a_2\delta-a_1 a_3=0. \tag{3.18}$$

We denote the solutions as δ_+ and δ_-:

$$\delta_\pm=\frac{a_2\pm(a_2^2+4a_1 a_3)^{1/2}}{2}. \tag{3.19}$$

We consider two situations.

(i) If

$$a_2^2+4a_1 a_3=0 \tag{3.20}$$

then

$$\delta_+=\delta_-=\tfrac{1}{2}a_2. \tag{3.21}$$

The solution to (3.17) is thus

$$u=\text{constant}\times(1-\tfrac{1}{2}a_2\lambda)\exp(\tfrac{1}{2}a_2\lambda). \tag{3.22}$$

Substituting this result into (3.16), (3.13) and (3.14) and employing the initial condition (3.15), we obtain the following result:

$$c_1(\lambda)=-\frac{a_2^2}{4a_3}\frac{\lambda}{(1-\frac{1}{2}a_2\lambda)} \tag{3.23a}$$

$$c_2(\lambda)=-2\ln(1-\tfrac{1}{2}a_2\lambda) \tag{3.23b}$$

$$c_3(\lambda)=\frac{2a_3}{a_2}\frac{1}{(1-\frac{1}{2}a_2\lambda)}. \tag{3.23c}$$

† It is clear that our method works even if the restriction is relaxed.

(ii) If

$$a_2^2+4a_1a_3\neq 0 \tag{3.24}$$

then

$$\delta_+\neq\delta_- \tag{3.25}$$

and so we have

$$u=C_1\exp(\delta_+\lambda)+C_2\exp(\delta_-\lambda) \tag{3.26}$$

where C_1 and C_2 are constants. This will lead to the results

$$c_1(\lambda)=\frac{\delta_+\delta_-}{a_3}\frac{\exp(\delta_+\lambda)-\exp(\delta_-\lambda)}{\delta_-\exp(\delta_+\lambda)-\delta_+\exp(\delta_-\lambda)} \tag{3.27a}$$

$$c_2(\lambda)=a_2\lambda-2\delta_+\delta_-\int_0^\lambda\frac{\exp(\delta_+\lambda')-\exp(\delta_-\lambda')}{\delta_-\exp(\delta_+\lambda')-\delta_+\exp(\delta_-\lambda')}\,\mathrm{d}\lambda' \tag{3.27b}$$

$$c_3(\lambda)=a_3\int_0^\lambda\exp(c_2(\lambda'))\,\mathrm{d}\lambda'. \tag{3.27c}$$

So the exponential operator in (3.7) is uncoupled into the product form

$$\exp[\lambda(a_1\hat{J}_++a_2\hat{J}_0+a_3\hat{J}_-)]=\exp(c_1(\lambda)\hat{J}_+)\exp(c_2(\lambda)\hat{J}_0)\exp(c_3(\lambda)\hat{J}_-) \tag{3.28}$$

with $c_i(\lambda)$ given by (3.23) or (3.27). By putting $\lambda=1$ in (3.28), we arrive at the following result:

$$\exp(a_1\hat{J}_++a_2\hat{J}_0+a_3\hat{J}_-)=\exp(c_1(1)\hat{J}_+)\exp(c_2(1)\hat{J}_0)\exp(c_3(1)\hat{J}_-). \tag{3.29}$$

To check the validity of (3.29), we shall derive the Baker-Campbell-Hausdorff formula (3.1) based on (3.29). Now we begin with

$$\hat{S}_1=\exp[\alpha\hat{J}_+-\alpha^*\hat{J}_-] \tag{3.30}$$

where α is expressible as

$$\alpha=-\tfrac{1}{2}\vartheta\exp(-\mathrm{i}\varphi) \tag{3.31}$$

in which ϑ, φ are real quantities.

Comparing (3.30) with (3.6), we have

$$a_1=\alpha \tag{3.32}$$

$$a_2=0 \tag{3.33}$$

$$a_3=-\alpha^*. \tag{3.34}$$

Now a_2 and a_1a_3 $(=-|\alpha|^2)$ are real and so the previous result (3.29) can be employed. One can then see that (3.24) is valid, i.e.

$$a_2+4a_1a_3=-4|\alpha|^2\neq 0. \tag{3.35}$$

By substituting (3.32)-(3.34) into (3.27) and after some manipulation, we readily obtain

$$\hat{S}_1=\exp[\alpha\hat{J}_+-\alpha^*\hat{J}_-]=\exp(c_1(1)\hat{J}_+)\exp(c_2(1)\hat{J}_0)\exp(c_3(1)\hat{J}_-) \tag{3.36}$$

with $c_i(\lambda)$ given by

$$c_1(\lambda)=-\exp(-\mathrm{i}\varphi)\tan\tfrac{1}{2}\vartheta\lambda \tag{3.37a}$$

$$c_2(\lambda)=-2\ln(\cos\tfrac{1}{2}\vartheta\lambda) \tag{3.37b}$$

$$c_3(\lambda)=\exp(\mathrm{i}\varphi)\tan\tfrac{1}{2}\vartheta\lambda. \tag{3.37c}$$

Putting $\lambda = 1$ in (3.37) will give (3.3)-(3.5) and (3.36) is then just the well known Baker-Campbell-Hausdorff formula for the SU(2) Lie group.

It should be remarked that, when a_2 and $a_1 a_3$ are complex quantities, equations (3.12)-(3.14) can also be solved. Our restriction for real a_2 and $a_1 a_3$ are only for the sake of simplicity.

Finally we would like to stress that the method for uncoupling exponential operators developed above is not restricted to the symmetry group SU(2). It can in principle be generalised to higher symmetry groups. Of course, in that case a larger set of differential equations have to be solved in order to obtain useful results.

4. Harmonic oscillator with varying mass

4.1. Evolution operator

We shall in this section employ the result in § 2 to study a harmonic oscillator with a varying mass. The general expression for the Hamiltonian of this oscillator is

$$\hat{H}(t) = \frac{\hat{p}^2}{2M(t)} + \tfrac{1}{2}M(t)\omega^2\hat{q}^2 \tag{4.1}$$

where $M(t)$ is the mass of the oscillator and is time dependent. Historically the above Hamiltonian has been employed to discuss dissipative systems and to describe damped oscillators of constant mass [8-13]. However this description leads to an unphysical result that seems to violate the uncertainty relation [8, 9]. This obscurity arises because the above Hamiltonian cannot describe a quantum damped oscillator of constant mass [14-16]. In fact the Hamiltonian represents an oscillator of variable mass, whose classical behaviour is identical to that of a damped oscillator of constant mass [14].

The Hamiltonian (4.1) has been investigated by a number of authors [17-25]. In the following we shall tackle this quantum problem by the evolution operator method developed in § 2. To start with, we rewrite the Hamiltonian (4.1) in the following form:

$$\hat{H}(t) = a_1(t)\hat{J}_+ + a_2(t)\hat{J}_0 + a_3(t)\hat{J}_- \tag{4.2}$$

where

$$\hat{J}_+ = \frac{1}{2\hbar}\hat{q}^2 \tag{4.3a}$$

$$\hat{J}_- = \frac{1}{2\hbar}\hat{p}^2 \tag{4.3b}$$

$$\hat{J}_0 = \frac{\mathrm{i}}{4\hbar}(\hat{p}\hat{q} + \hat{q}\hat{p}) \tag{4.3c}$$

and

$$a_1(t) = \hbar M(t)\omega^2 \tag{4.4a}$$

$$a_2(t) = 0 \tag{4.4b}$$

$$a_3(t) = \frac{\hbar}{M(t)}. \tag{4.4c}$$

In view of equations (2.7a), (2.10) and (2.11), we can represent the evolution operator for the above Hamiltonian as follows:

$$\hat{U}(t,0)=\exp(c_1(t)\hat{J}_+)\exp(c_2(t)\hat{J}_0)\exp(c_3(t)\hat{J}_-) \tag{4.5}$$

with $c_i(t)$ given by

$$c_1(t)=\mathrm{i}M(t)\frac{\partial}{\partial t}[\ln u(t)] \tag{4.6a}$$

$$c_2(t)=-2\ln\frac{u(t)}{u(0)} \tag{4.6b}$$

$$c_3(t)=-\mathrm{i}u^2(0)\int_0^t\frac{\mathrm{d}t'}{M(t')u^2(t')} \tag{4.6c}$$

in which $u(t)$ satisfies the following differential equation:

$$\ddot{u}+\gamma_0(t)\dot{u}+\omega^2u=0 \tag{4.7}$$

and

$$\gamma_0(t)=\frac{\partial}{\partial t}[\ln M(t)]. \tag{4.8}$$

It is interesting to see that the expression for $c_i(t)$ is related to the solution of the differential equation (4.7) which is just the equation of motion for a classical damped oscillator of constant mass.

4.2. *Coherence and squeezing property of the wavefunction*

Having found the evolution operator in (4.5), we can look at the evolution of a coherent state and discuss its squeezing and coherence property.

Suppose we start with a coherent state at $t=0$:

$$|\Phi(0)\rangle=|\alpha\rangle. \tag{4.9}$$

The wavefunction at any later time will be represented by

$$|\Phi(t)\rangle=\hat{U}(t,0)|\alpha\rangle. \tag{4.10}$$

Performing the usual quantisation procedure at $t=0$

$$\hat{a}=\frac{1}{(2m\hbar\omega)^{1/2}}(m\omega\hat{q}+\mathrm{i}\hat{p}) \tag{4.11}$$

in which

$$m=M(0) \tag{4.12}$$

we can define a new operator $\hat{A}$ as

$$\hat{A}=\hat{U}(t,0)\hat{a}\hat{U}^+(t,0). \tag{4.13}$$

It is easy to see that the wavefunction $|\Phi(t)\rangle$ is a coherent state with respect to this new operator

$$\hat{A}|\Phi(t)\rangle=\alpha|\Phi(t)\rangle. \tag{4.14}$$

Using (4.5) and (4.11), it can be shown that the original operator $\hat{a}$ is related to the new operator $\hat{A}$ by a Bogoliubov transformation

$$\hat{A} = \eta_1 \hat{a} - \eta_2 \hat{a}^+ \tag{4.15}$$

with

$$|\eta_1|^2 - |\eta_2|^2 = 1 \tag{4.16}$$

where η_1 and η_2 are given by

$$\eta_1 = \frac{\exp(-\frac{1}{2}c_2)}{2}[1 + c_1 c_3 + \exp(c_2) - c_1/m\omega - m\omega c_3] \tag{4.17a}$$

$$\eta_2 = \frac{\exp(-\frac{1}{2}c_2)}{2}[1 - c_1 c_3 - \exp(c_2) + c_1/m\omega - m\omega c_3]. \tag{4.17b}$$

The result (4.14), together with (4.15), implies that the wavefunction $|\Phi(t)\rangle$ is a squeezed state. So the wavefunction starts as a coherent state at time $t = 0$ and evolves as a squeezed state at a later time.

To discuss the squeezing property of this wavefunction, it is convenient to define the following two operators:

$$\hat{x}_1 = \tfrac{1}{2}(\hat{a} + \hat{a}^+) \tag{4.18a}$$

$$\hat{x}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^+). \tag{4.18b}$$

Using the evolution operator in (4.5), it can be shown that the expectation values of these two operators with respect to the wavefunction (4.10) is given by

$$\begin{aligned}\langle \hat{x}_1 \rangle &= \langle \Phi(t)|\hat{x}_1|\Phi(t)\rangle \\ &= \frac{\exp(-\frac{1}{2}c_2)}{2}[(1 + m\omega c_3)\alpha + (1 - m\omega c_3)\alpha^*]\end{aligned} \tag{4.19}$$

$$\begin{aligned}\langle \hat{x}_2 \rangle = \frac{\exp(-\frac{1}{2}c_2)}{2i}\{&[c_1 c_3 + \exp(c_2) + c_1/m\omega]\alpha \\ &- [c_1 c_3 + \exp(c_2) - c_1/m\omega]\alpha^*\}.\end{aligned} \tag{4.20}$$

The corresponding fluctuations in $\hat{x}_1$ and $\hat{x}_2$ will then be

$$\Delta x_1^2 = \tfrac{1}{4}\exp(-c_2)(1 - m^2\omega^2 c_3^2) \tag{4.21}$$

$$\Delta x_2^2 = \tfrac{1}{4}\exp(-c_2)\{[c_1 c_3 + \exp(c_2)]^2 - c_1^2/m^2\omega^2\}. \tag{4.22}$$

Immediately we see that

$$\Delta x_1 \propto \exp(-\tfrac{1}{2}c_2) \tag{4.23}$$

$$\Delta x_2 \propto \exp(\tfrac{1}{2}c_2). \tag{4.24}$$

So we obtain squeezing in the fluctuation of one operator in the expanse of an increase in the fluctuation of the other operator. Thus the squeezing property of $|\Phi(t)\rangle$ is apparent here.

4.3. Model Hamiltonian

We shall in this section consider a specific model of mass variation for the oscillator. We assume that the mass of the oscillator is exponentially increasing (or decreasing) in time:

$$M(t) = m \exp(\gamma t). \tag{4.25}$$

The resulting Hamiltonian corresponds to the Kanai Hamiltonian [8]. This specific model has been studied by a number of authors [17, 18, 20, 22, 25] using the transformation method or the Green function approach. Here we wish to demonstrate that our evolution operator method can yield the exact closed-form solution to the problem.

First we note that we have to find the solution to the differential equation (4.7). From (4.25), $\gamma_0(t)$ in (4.8) becomes

$$\gamma_0(t) = \gamma \tag{4.26}$$

so that the differential equation (4.7) now corresponds to the equation of motion for a constant-mass oscillator under constant damping

$$\ddot{u} + \gamma \dot{u} + \omega^2 u = 0. \tag{4.27}$$

This equation can be solved readily and we obtain the following results for the coefficients $c_i(t)$ appearing in the evolution operator (4.5).

(i) When $\gamma^2 < 4\omega^2$ (underdamping)

$$c_1(t) = -\mathrm{i}m\omega^2 \frac{\exp(\gamma t)}{\frac{1}{2}\gamma + \frac{1}{2}\xi \cot \frac{1}{2}\xi t} \tag{4.28a}$$

$$c_2(t) = \gamma t - 2 \ln[(\gamma/\xi) \sin \tfrac{1}{2}\xi t + \cos \tfrac{1}{2}\xi t] \tag{4.28b}$$

$$c_3(t) = -\frac{\mathrm{i}}{m} \frac{1}{\frac{1}{2}\gamma + \frac{1}{2}\xi \cot \frac{1}{2}\xi t} \tag{4.28c}$$

where

$$\xi = (4\omega^2 - \gamma^2)^{1/2}. \tag{4.29}$$

(ii) When $\gamma^2 = 4\omega^2$ (critical damping)

$$c_1(t) = -\mathrm{i}m\omega^2 \frac{t \exp(\gamma t)}{1 + \frac{1}{2}\gamma t} \tag{4.30a}$$

$$c_2(t) = \gamma t - 2 \ln(1 + \tfrac{1}{2}\gamma t) \tag{4.30b}$$

$$c_3(t) = -\frac{\mathrm{i}}{m} \frac{t}{1 + \frac{1}{2}\gamma t}. \tag{4.30c}$$

(iii) When $\gamma^2 > 4\omega^2$ (overdamping)

$$c_1(t) = -\mathrm{i}m\omega^2 \frac{\exp(\gamma t)}{\frac{1}{2}\gamma + \frac{1}{2}\zeta \coth \frac{1}{2}\zeta t} \tag{4.31a}$$

$$c_2(t) = \gamma t - 2 \ln[(\gamma/\zeta) \sinh \tfrac{1}{2}\zeta t + \cosh \tfrac{1}{2}\zeta t] \tag{4.31b}$$

$$c_3(t) = -\frac{\mathrm{i}}{m} \frac{1}{\frac{1}{2}\gamma + \frac{1}{2}\zeta \coth \frac{1}{2}\zeta t} \tag{4.31c}$$

where

$$\zeta = (\gamma^2 - 4\omega^2)^{1/2}. \tag{4.32}$$

Now the evolution operator is completely determined by the above expressions. Hence we can readily evaluate expectation values of the energy, position and momentum.

First of all we look at the expectation value of the energy. We assume that the initial wavefunction is a coherent state (4.9)

$$|\Phi(0)\rangle = |\alpha\rangle$$

in which we represent α as

$$\alpha = (n_0)^{1/2} \exp(\mathrm{i}\varphi). \tag{4.33}$$

Making use of the results in (4.28)-(4.32), the expectation value of the energy can be found explicitly as follows.

(i) When $\gamma^2 < 4\omega^2$

$$E = \hbar\omega\left[(n_0+\tfrac{1}{2})\left(1+2\frac{\gamma^2}{\xi^2}\sin^2\tfrac{1}{2}\xi t\right) + n_0\frac{\gamma}{\xi}\sin \xi t \cos 2\varphi + n_0\frac{\gamma\omega}{\xi^2}\sin^2\tfrac{1}{2}\xi t \sin 2\varphi\right]. \tag{4.34}$$

(ii) When $\gamma^2 = 4\omega^2$

$$E = \hbar\omega[(n_0+\tfrac{1}{2})(1+2\omega^2 t^2) + n_0\gamma t \cos 2\varphi + n_0\gamma\omega t^2 \sin 2\varphi]. \tag{4.35}$$

(iii) When $\gamma^2 > 4\omega^2$

$$E = \hbar\omega\left[(n_0+\tfrac{1}{2})\left(1+2\frac{\gamma^2}{\xi^2}\sinh^2\tfrac{1}{2}\zeta t\right) + n_0\frac{\gamma}{\zeta}\sinh \zeta t \cos 2\varphi + n_0\frac{\gamma\omega}{\zeta^2}\sinh^2\tfrac{1}{2}\zeta t \sin 2\varphi\right]. \tag{4.36}$$

We readily observe that, for $\gamma^2 < 4\omega^2$, the energy oscillates periodically, while for $\gamma^2 \geqslant 4\omega^2$, the energy will not oscillate any more. In figure 1, we plot the energy against time to illustrate the above feature.

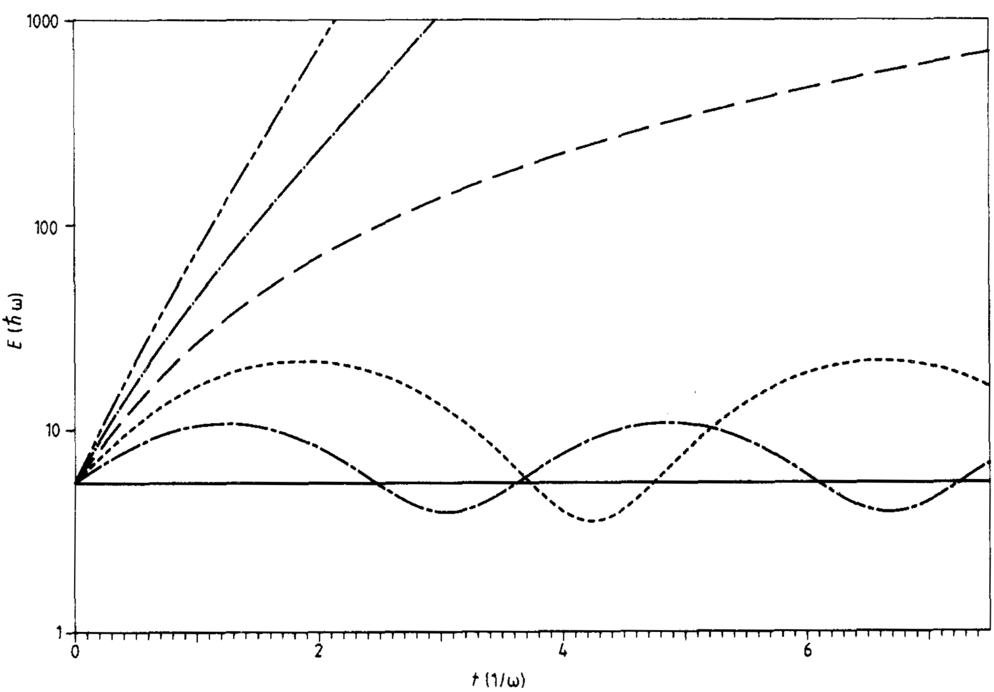

Figure 1. Time variation of the energy (with $n_0 = 5$ and $\varphi = 0$) for $\gamma/2\omega$ taking the values 0 (——), 0.5 (— — —), 0.75 (- - -), (— — —), 1.25 (— · —) and 1.5 (— - - —).

Here, if we denote the energy as

$$E = E(\gamma, \varphi) \tag{4.37}$$

then it will have the following properties:

$$E(\gamma, \varphi) = E(\gamma, \varphi + \pi) \tag{4.38}$$

$$E(-\gamma, \varphi) = E(\gamma, \varphi + \tfrac{1}{2}\pi). \tag{4.39}$$

Equation (4.38) illustrates the symmetry property of the energy with respect to the phase angle φ while (4.39) relates the energy for a positive γ oscillator with that of a negative γ oscillator.

Now we come to look at the expectation values of the position and momentum for the mass-varying oscillator. We see from (4.18) that

$$\hat{q} = \left(\frac{2\hbar}{m\omega}\right)^{1/2} \hat{x}_1 \tag{4.40}$$

$$\hat{p} = (2\hbar m\omega)^{1/2} \hat{x}_2. \tag{4.41}$$

So with the help of the results (4.19)-(4.22), the expectation values for $\hat{p}$ and $\hat{q}$, together with their fluctuations, can be evaluated and are given below.

(i) When $\gamma^2 < 4\omega^2$

$$\bar{q} = \langle\Phi(t)|\hat{q}|\Phi(t)\rangle$$

$$= \left(\frac{2\hbar n_0}{m\omega}\right)^{1/2} \exp(-\tfrac{1}{2}\gamma t)\frac{2}{\xi}\sin(\tfrac{1}{2}\xi t)[(\tfrac{1}{2}\xi \cot\tfrac{1}{2}\xi t + \tfrac{1}{2}\gamma)\cos\varphi + \omega\sin\varphi] \tag{4.42}$$

$$\Delta q^2 = \frac{\hbar}{2m\omega}\exp(-\gamma t)\left(\frac{2}{\xi}\sin\tfrac{1}{2}\xi t\right)^2[\omega^2 + (\tfrac{1}{2}\gamma + \tfrac{1}{2}\xi\cot\tfrac{1}{2}\xi t)^2] \tag{4.43}$$

$$\bar{p} = (2n_0\hbar m\omega)^{1/2}\exp(\tfrac{1}{2}\gamma t)(2/\xi)\sin\tfrac{1}{2}\xi t[(\tfrac{1}{2}\xi\cot\tfrac{1}{2}\xi t - \tfrac{1}{2}\gamma)\sin\varphi - \omega\cos\varphi] \tag{4.44}$$

$$\Delta p^2 = \tfrac{1}{2}\hbar m\omega\exp(\gamma t)[(2/\xi)\sin\tfrac{1}{2}\xi t]^2[\omega^2 + (\tfrac{1}{2}\gamma - \tfrac{1}{2}\xi\cot\tfrac{1}{2}\xi t)^2]. \tag{4.45}$$

So the uncertainty relation is

$$\Delta p\Delta q = \tfrac{1}{2}\hbar[(2/\xi)\sin\tfrac{1}{2}\xi t]^2[(\omega^2 + \tfrac{1}{4}\gamma^2 - \tfrac{1}{4}\xi^2\cot^2\tfrac{1}{2}\xi t)^2 + \omega^2\xi^2\cot^2\tfrac{1}{2}\xi t]^{1/2}. \tag{4.46}$$

(ii) When $\gamma^2 = 4\omega^2$

$$\bar{q} = \left(\frac{2\hbar n_0}{m\omega}\right)^{1/2}\exp(-\tfrac{1}{2}\gamma t)[(1 + \tfrac{1}{2}\gamma t)\cos\varphi + \omega t\sin\varphi] \tag{4.47}$$

$$\Delta q^2 = \frac{\hbar}{2m\omega}\exp(-\gamma t)(1 + \gamma t + 2\omega^2 t^2) \tag{4.48}$$

$$\bar{p} = (2n_0\hbar m\omega)^{1/2}\exp(\tfrac{1}{2}\gamma t)[(1 - \tfrac{1}{2}\gamma t)\sin\varphi - \omega t\cos\varphi] \tag{4.49}$$

$$\Delta p^2 = \tfrac{1}{2}\hbar m\omega\exp(\gamma t)(1 - \gamma t + 2\omega^2 t^2) \tag{4.50}$$

$$\Delta p\Delta q = \tfrac{1}{2}\hbar(1 + 4\omega^4 t^4)^{1/2}. \tag{4.51}$$

(iii) When $\gamma^2 > 4\omega^2$

$$\bar{q} = \left(\frac{2\hbar n_0}{m\omega}\right)^{1/2}\exp(-\tfrac{1}{2}\gamma t)\frac{2}{\zeta}\sinh\tfrac{1}{2}\zeta t[(\tfrac{1}{2}\zeta\coth\tfrac{1}{2}\zeta t + \tfrac{1}{2}\gamma)\cos\varphi + \omega\sin\varphi] \tag{4.52}$$

$$\Delta q^2 = \frac{\hbar}{2m\omega} \exp(-\gamma t)[(2/\zeta) \sinh \tfrac{1}{2}\zeta t]^2[\omega^2 + (\tfrac{1}{2}\gamma + \tfrac{1}{2}\zeta \coth \tfrac{1}{2}\zeta t)^2] \tag{4.53}$$

$$\bar{p} = (2n_0\hbar m\omega)^{1/2} \exp(\tfrac{1}{2}\gamma t) \times (2/\zeta) \sinh \tfrac{1}{2}\zeta t[(\tfrac{1}{2}\zeta \coth \tfrac{1}{2}\zeta t - \tfrac{1}{2}\gamma) \sin \varphi - \omega \cos \varphi] \tag{4.54}$$

$$\Delta p^2 = \tfrac{1}{2}\hbar m\omega \exp(\gamma t)[(2/\zeta) \sinh \tfrac{1}{2}\zeta t]^2[\omega^2 + (\tfrac{1}{2}\gamma - \tfrac{1}{2}\zeta \coth \tfrac{1}{2}\zeta t)^2] \tag{4.55}$$

$$\Delta p \Delta q = \tfrac{1}{2}\hbar[(2/\zeta) \sinh \tfrac{1}{2}\zeta t]^2[(\omega^2 + \tfrac{1}{4}\gamma^2 - \tfrac{1}{4}\zeta^2 \coth^2 \tfrac{1}{2}\zeta t)^2 + \omega^2\zeta^2 \coth^2 \tfrac{1}{2}\zeta t]^{1/2}. \tag{4.56}$$

In the above results, we immediately see that all the fluctuations Δp^2, Δq^2 and $\Delta p \Delta q$ are independent of n_0 and φ. In fact, Δp and Δq give the magnitude of the vacuum fluctuations in $\hat{p}$ and $\hat{q}$. In figures 2 and 3, we plot the time variation of the fluctuations Δq^2 and Δp^2, respectively. These diagrams clearly show that there is squeezing in the fluctuation of $\hat{q}$, together with an increase in the fluctuation in $\hat{p}$. However the uncertainty relation

$$\Delta p \Delta q \geqslant \tfrac{1}{2}\hbar \tag{4.57}$$

is satisfied at all times. This is depicted in figure 4.

Now let us turn to look at the expectation value of the position of the oscillator. Since we start with a coherent state, this coherent state will generate the classical behaviour for the mass-varying oscillator. In figure 5 we give the plot of the time variation of the position of the oscillator. It clearly shows that the motion of this mass-varying oscillator resembles that of a classical damped harmonic oscillator. In this case, γ in (4.25) appears to take the role of a damping coefficient.

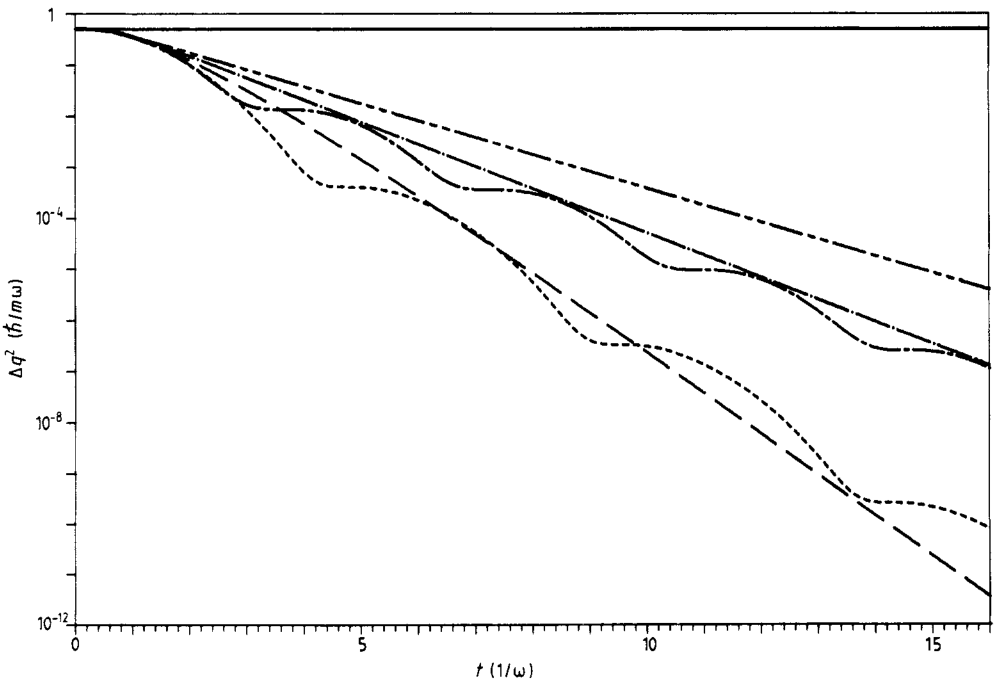

Figure 2. Time variation of the variance in $q, \Delta q^2$. (The same symbols are used as in figure 1.)

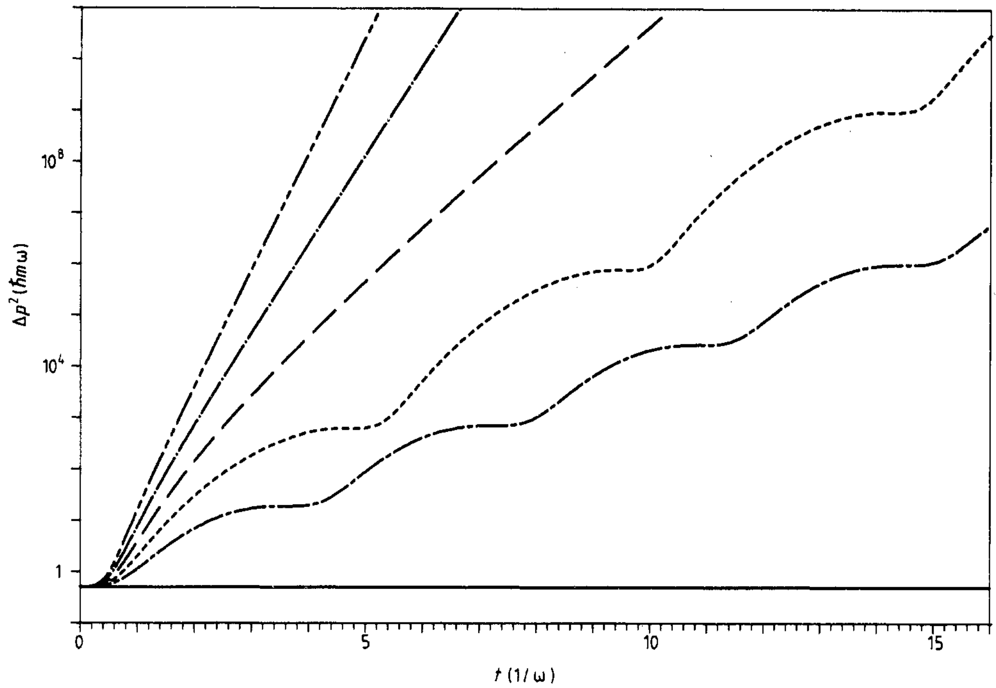

Figure 3. Time variation of the variance in p, Δp^2. (The same symbols are used as in figure 1.)

Finally if we represent $\bar{p}$ and $\bar{q}$ by

$$\bar{p} = \bar{p}(\gamma, \varphi) \tag{4.58}$$

$$\bar{q} = \bar{q}(\gamma, \varphi) \tag{4.59}$$

then the following properties for $\bar{p}$ and $\bar{q}$ are observed:

$$\bar{q}(\gamma, \varphi) = -\bar{q}(\gamma, \varphi + \pi) \tag{4.60}$$

$$\bar{p}(\gamma, \varphi) = -\bar{p}(\gamma, \varphi + \pi) \tag{4.61}$$

$$\bar{q}(-\gamma, \varphi) = \frac{1}{m\omega}\,\bar{p}(\gamma, \varphi + \tfrac{1}{2}\pi) \tag{4.62}$$

$$\bar{p}(-\gamma, \varphi) = m\omega\bar{q}(\gamma, \varphi + \tfrac{3}{2}\pi). \tag{4.63}$$

5. Conclusion

We have developed explicitly a method to find the evolution operator for a Hamiltonian expressible in the form (1.5). Our procedure is to assume the form for the evolution operator (as in (2.4)) and then construct a set of ordinary differential equations (i.e. (2.7)) for the coefficients. The critical step is basically in the choice of the form (2.4) for the evolution operator. This method can, in principle, be applied to other Hamiltonian systems consisting of group generators of symmetries other than the SU(2) symmetry assumed here.

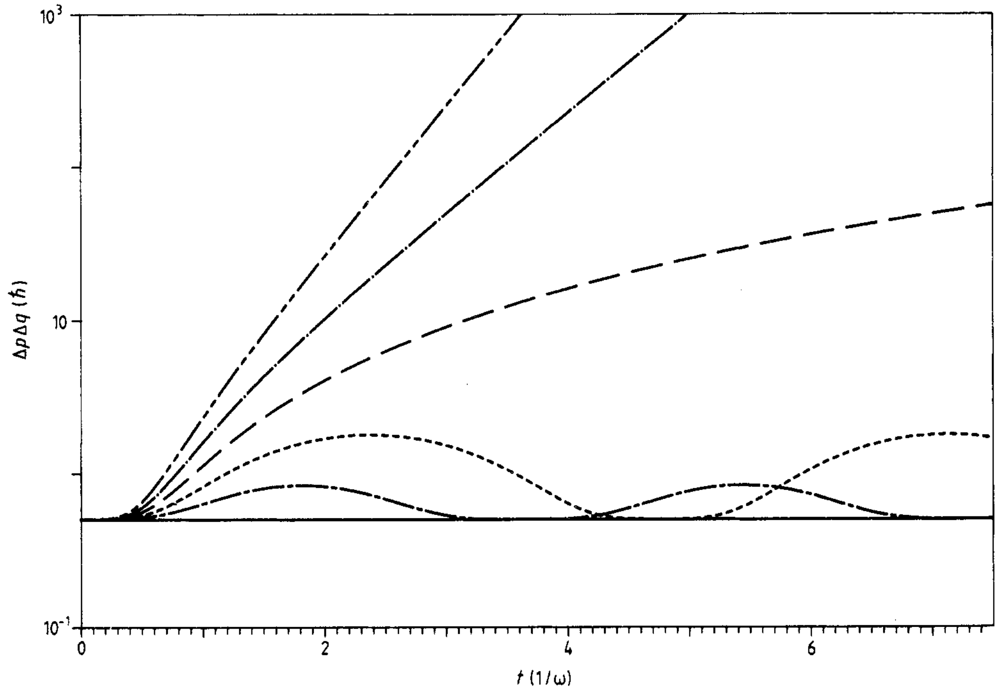

Figure 4. Time variation of the product uncertainty $\Delta p \Delta q$. (The same symbols are used as in figure 1.)

As an application of our methodology, we have demonstrated a way of uncoupling an exponential operator. Our method does not require that the exponential operator be unitary. Thus this disentangling technique can be applied to a larger range of exponential operators than those applicable by the Baker–Campbell–Hausdorff formula in (3.1). This BCH formula is deduced from our result as a special example.

To apply the result in § 2 further, we consider the problem of a time-dependent harmonic oscillator with a varying mass. The evolution operator is explicitly found†. Now, using our derived evolution operator, we can show that a coherent state will evolve as a squeezed state. When considering a specific type of mass variation, the expectation value for the energy, position and momentum are found. Fluctuations in the position and momentum are also evaluated. This method enables us to perform explicit numerical computation. Our analysis shows that there is a close relationship between our time-dependent oscillator and a damped harmonic oscillator. Besides it is shown that a harmonic oscillator with negative γ value can be related simply to those with positive γ ones (cf (4.39), (4.62) and (4.63)). We have presented a numerical analysis for positive γ only. It is seen explicitly that there is squeezing in the fluctuation in the position. However, for oscillators with negative γ, there will be squeezing in the momentum instead.

Finally we come back to the evolution operator. At the beginning, we employed the form (1.4) of the evolution operator. However, this representation is local in time, namely valid in a range of time intervals. Such a restriction can be seen to arise from

† In a recent paper by Fernández [26], the evolution operator is found by another method. However, the explicit form is not given.

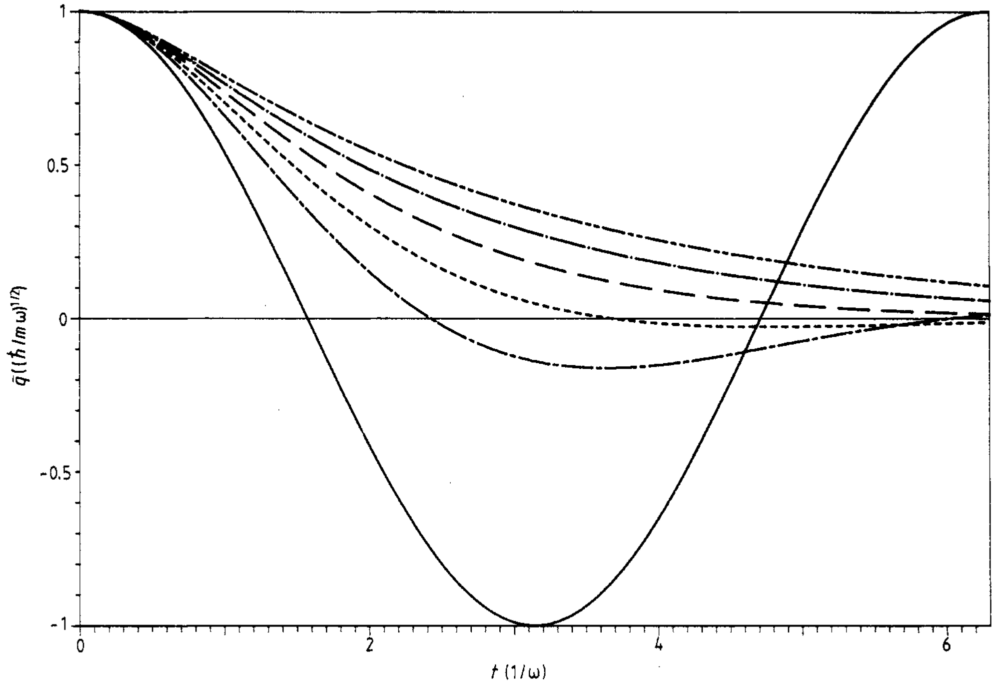

Figure 5. Variation of the expectation value of position $\bar{q}$ at different times for $n_0 = 0.5$ and $\varphi = 0$. (The same symbols are used as in figure 1.)

the form of the coefficient $c_2(t)$ appearing in (4.28b), (4.30b) and (4.31b) in our specific problem. As we know, the expression inside the natural logarithmic function in c_2 should be positive. However, as we noted, in all the expressions we have derived, c_2 only appears in the following form: $\exp(\pm c_2)$. In this way the restriction due to the natural logarithmic function is in some sense relaxed by the exponential function. So, on the whole, we encounter no restriction on time and it appears that the results obtained so far are valid for all time.

Finally we note that the method developed so far can, in principle, be extended to other types of Hamiltonian.

References

[1] Dyson F J 1949 *Phys. Rev.* **75** 1736-55
[2] Lam C C and Fung P C W 1983 *Phys. Rev.* A **27** 1760-78
[3] Magnus W 1954 *Commun. Pure Appl. Math.* **7** 649-73
[4] Wei J and Norman E 1963 *J. Math. Phys.* **4** 575-81
[5] Perelomov A M 1977 *Sov. Phys.-Usp.* **20** 703-20
Gilmore R 1974 *Lie Groups, Lie Algebras, and Some of Their Applications* (New York: Wiley)
[6] Truax D R 1985 *Phys. Rev.* D **31** 1988-91
[7] Witschel W 1985 *Phys. Lett.* **111A** 383-8
[8] Kanai E 1948 *Prog. Theor. Phys.* **3** 440-2
[9] Brittin W E 1950 *Phys. Rev.* **77** 396-7
[10] Hasse R W 1975 *J. Math. Phys.* **16** 2005-11
[11] Messer J 1979 *Acta Phys. Aust.* **50** 75-91
[12] Remaund B and Hernandez E S 1980 *J. Phys. A: Math. Gen.* **13** 2013-28

[13] Dekker H 1981 *Phys. Rep.* **80** 1-112
[14] Greenberger D M 1979 *J. Math. Phys.* **20** 762-70
[15] Greenberger D M 1979 *J. Math. Phys.* **20** 771-80
[16] Ray J R 1979 *Am. J. Phys.* **47** 626-9
[17] Landovitz L F, Levine A M and Schreiber W M 1979 *Phys. Rev.* A **20** 1162-8
[18] Colegrave R K and Abdalla M S 1981 *J. Phys. A: Math. Gen.* **14** 2269-80
[19] Colegrave R K and Abdalla M S 1982 *J. Phys. A: Math. Gen.* **15** 1549-59
[20] Leach P G L 1983 *J. Phys. A: Math. Gen.* **16** 3261-9
[21] Colegrave R K and Abdalla M S 1983 *J. Phys. A: Math. Gen.* **16** 3805-15
[22] Abdalla M S and Colegrave R K 1984 *Lett. Nuovo Cimento* **39** 373-9
[23] Abdalla M S 1985 *Lett. Nuovo Cimento* **44** 482-6
[24] Cheng B K 1985 *Phys. Lett.* **113A** 293-6
[25] Abdalla M S 1986 *Phys. Rev.* A **33** 2870-6
[26] Fernández F M 1987 *J. Math. Phys.* **28** 2908